

Operator Algebra in Logarithmic Conformal Field Theory

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Abstract

For some time now, conformal field theories in two dimensions have been studied as integrable systems. Much of the success of these studies is related to the existence of an operator algebra of the theory. In this note, some of the extensions of this machinery to the logarithmic case are studied, and used. More precisely, from Möbius symmetry constraints, the generic three and four point functions of logarithmic quasiprimary fields are calculated in closed form for arbitrary Jordan rank. As an example, $c = 0$ disordered systems with non-degenerate vacua are studied. With the aid of two, three and four point functions, the operator algebra is obtained and associativity of the algebra studied.

1 Introduction

Since the studies of Belavin, Polyakov and Zamolodchikov [1] and Zamolodchikov and Zamolodchikov [15], Conformal Field Theory in two dimensions (2dCFT) has been heavily studied, in particular as an integrable quantum field theory. This means that, in principle, the model is exactly solvable, i.e. all of the correlation functions can be determined, and hence the S-matrix. Central to this idea is the notion of an ‘operator algebra’, where the operators of the theory are endowed with a product, so that a product of two operators may be realized as a linear combination of other operators in the theory. Thus, the calculation of an N -point function can be reduced to one of a linear combination of $N - 1$ point functions. Given a quantum field theory, one might ask - what is the operator algebra? In general, this is a difficult question to answer, but in 2dCFT, [1] demonstrated how to obtain it. The method depends crucially on the form of two, three and four point functions of ‘primary’ fields of the theory. Primary fields of the theory are fundamental fields of the theory, in the sense that all of the other fields can be obtained by action of the symmetry algebra on the primary fields.

Logarithmic conformal field theories (LCFTs) are a class of 2dCFT that have been studied heavily over the last 10 years. They differ from more traditional 2dCFTs by having logarithms in correlation functions, and having elements of the Cartan subalgebra which are non-diagonalizable - most notably the dilation operator L_0 is non-diagonalizable, and hence must be represented in terms of Jordan blocks. Due to these complexities, obtaining an understanding of the 2, 3, 4 point functions and the operator algebra has been tricky, although much work has been done towards this, including, [2][3][4][5][6][10][11][12][13].

In this note, using the constructions developed in [13] (which are reviewed in section 2), the three and four point functions of logarithmic quasiprimary fields are calculated (sections 3 and 4 respectively). It should be stressed that in the calculation of these correlators, no assumptions on the operator algebra are made, and no ansatz is used, all of which are often used in the literature. The calculations are valid for arbitrary Jordan block size, even when the different Jordan blocks inside the correlator are of different rank. The only input is the conformal Ward identities corresponding to the Möbius group, which are subsequently solved for, yielding general solutions.

In order to try and find an operator algebra using these 2, 3, 4 point functions, the case of $c = 0$ systems with non-degenerate vacua (that is, the vacuum does not have a logarithmic partner - for the degenerate vacua case, see [7][11]) is analyzed, which pertains to two dimensional systems with quenched disorder [8][9]. This is an unusual

LCFT, in that the vacuum does not belong to a Jordan block, whereas the stress energy tensor does. Indeed, there are questions as to whether or not such an unusual LCFT can be consistent. In section 5, it is found that from just assuming that the vacuum is non-degenerate, that $c = 0$, and that the stress energy tensor (a.k.a. the Virasoro generator) has a logarithmic partner (34), (35), with help from the three point function of section 3, one is, remarkably, able to find the entire operator algebra, which corresponds to the one found in [11]. In section 6, a partial study of associativity is then conducted, which, as is the norm in 2dCFT, comes down to studying four point functions. No inconsistencies are found.

2 Review of Logarithmic Primaries

Logarithmic conformal field theories are conformal field theories that are characterized by L_0 being non-diagonalizable and logarithms appearing in correlation functions. To this end, one can try and alter the definition of a primary field to accommodate the non-diagonal behaviour, and see if logarithms come out. The author should stress that it is not obvious to him if all logarithmic conformal field theories can be realized in this way.

Consider an action of the Virasoro algebra on fields $\phi_i(z)$, $i = 0 \dots N - 1$ given by $m \in \mathbb{Z}$,

$$[L_m, \phi_i(z)] = z^m(m+1)(h\phi_i(z) + \phi_{i+1}(z)) + z^{m+1}\partial\phi_i(z) \quad i = 0 \dots N-2 \quad (1)$$

$$[L_m, \phi_{N-1}(z)] = z^m(m+1)h\phi_{N-1}(z) + z^{m+1}\partial\phi_{N-1}(z). \quad (2)$$

Now, the ϕ_{i+1} term in (1) prevents this from being a collection of N primary fields of conformal weight h - indeed as it stands there is only one primary field. Acting on the vacuum $|0\rangle$, and considering $z = 0$, one finds

$$L_0|\phi_i\rangle = h|\phi_i\rangle + |\phi_{i+1}\rangle \text{ for } i = 0 \dots N-2, \quad L_0|\phi_{N-1}\rangle = h|\phi_{N-1}\rangle \quad (3)$$

and thus the primary field corresponds to the eigenvector of the Jordan block. In light of this, one can construct a vector $v_\phi(z)$ out of the $\phi_i(z)$, and rewrite (1)(2) as

$$[L_m, v(z)] = z^m(m+1)(h+J)v_\phi(z) + z^{m+1}\partial v_\phi(z) \quad (4)$$

where J is a rank N nilpotent matrix, that is satisfies $J^N = 0$, $J^{N-1} \neq 0$. Now, one might try to integrate up (4), to obtain a geometric object, and one finds that $v(z)$ can

be realized as a section of the formal rank n vector bundle whose transition functions are generated by dz^{h+J} (see [13] for more details). Now, given transition functions for a vector bundle, one is always free to rewrite everything in terms of G -bundles. In the case at hand, this translates to defining

$$\phi(z, J) := \sum_{i=0}^{N-1} \phi_i(z) J^{N-i-1} \quad (5)$$

in which case (4) reads

$$[L_m, \phi(z, J)] = z^m(m+1)(h+J)\phi(z, J) + z^{m+1}\partial\phi(z, J) \quad (6)$$

and defines a logarithmic primary field ϕ of weight h and rank N . If (6) only holds for $m \in \{-1, 0, 1\}$, then ϕ is a logarithmic quasiprimary field. It should be emphasized that (6), (4) and the pair (1)(2) are equivalent ways of describing the same thing. Whilst (4) might be a more convenient realization when studying representation theory, (6) is more convenient for studying the operator algebra and correlation functions, and hence will be used here.

As is usual in conformal field theory, one can restrict to $m \in \{1, 0, -1\}$ to obtain the action under the Lie algebra of the Möbius group. Since the $L_0, L_{\pm 1}$ annihilate the vacuum, these can be used to give readily solvable Ward identities for the correlation functions. For example, one has on the two point function $\langle \phi(z, J) \otimes \psi(w, K) \rangle$,

$$\langle [L_m, \phi(z, J)] \otimes \psi(w, K) \rangle + \langle \phi(z, J) \otimes [L_m, \psi(w, K)] \rangle = 0 \quad (7)$$

which can be solved [13] to yield

$$\langle \phi(z, J) \otimes \psi(w, K) \rangle = \mathbf{C}(J, K)(z-w)^{-2(\mathbb{I} \otimes \mathbb{I} h_1 + J \otimes \mathbb{I})} \quad (8)$$

where, for the two point function to be non-zero, one must have the conformal weights of ϕ and ψ identical, i.e. $h_1 - h_2 = 0$, as well as $(J - K)\mathbf{C} = 0$, where \mathbf{C} is a ‘function’ of the J, K , i.e. has an expansion

$$\mathbf{C} = \sum_{i=0, j=0}^{N-1, M-1} C_{i,j} J^i \otimes K^j. \quad (9)$$

Note, for the particular case of $N = M = 2$, that is $J^2 = 0 = K^2$, one has the two point

function (surpressing tensor products)

$$\langle \phi(z, J) \psi(w, K) \rangle = (z - w)^{-2h_1} \left((J + K)a + JK(b - 2a \log(z - w)) \right) \quad (10)$$

where a and b are arbitrary, yielding the logarithms, as promised. For higher rank Jordan blocks, the solution, when expressed in components, can become quite unwieldy, with powers of logarithms all over the place. For the remainder of this note, the tensor products will be surpressed for clarity.

3 Three point function

Consider the three point function

$$\langle \phi_1(x, J) \phi_2(y, K) \phi_3(z, L) \rangle = \mathbf{f}(x, y, z, J, K, L) \quad (11)$$

where $J^M = 0$, $J^{M-1} \neq 0$, $K^N = 0$, $K^{N-1} \neq 0$, $L^P = 0$ and $L^{P-1} \neq 0$ for some $M, N, P \in \mathbb{Z}$ with $M, N, P \geq 2$. For the purposes of this calculation, the co-ordinates

$$t = x - y, \quad u = y - z, \quad v = z + x \quad (12)$$

are useful. The L_{-1} condition then becomes

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \mathbf{f} = 2 \frac{\partial}{\partial v} \mathbf{f} = 0. \quad (13)$$

Hence $\mathbf{f} = \mathbf{f}(t, u, J, K, L)$. The L_0 and L_1 conditions then read

$$\left(h_1 + h_2 + h_3 + J + K + L + t \frac{\partial}{\partial t} + u \frac{\partial}{\partial u} \right) \mathbf{f} = 0 \quad (14)$$

$$\begin{aligned} & \left((v + u + t)(h_1 + J) + (v + u - t)(h_2 + K) + (v - u - t)(h_3 + L) \right. \\ & \quad \left. + (u + v)t \frac{\partial}{\partial t} + (v - t)u \frac{\partial}{\partial u} \right) \mathbf{f} = 0 \end{aligned} \quad (15)$$

respectively. Now, instead of (14) and (15), one could consider (16) = $(t - v)(14) + (15)$ and (17) = $-(u + v)(14) + (15)$. Since the transformation is invertible, the conditions (16) and (17) are equivalent to the conditions (14) and (15). Hence, one has

$$\left((2t + u)(h_1 + J) + u(h_2 + K) - u(h_3 + L) + (u + t)t \frac{\partial}{\partial t} \right) \mathbf{f} = 0 \quad (16)$$

$$\left(t(h_1 + J) - t(h_2 + K) - (2u + t)(h_3 + L) - (u + t)u \frac{\partial}{\partial u}\right) \mathbf{f} = 0. \quad (17)$$

On expanding \mathbf{f} in J, K, L , (16) gives rise to $M \times N \times P$ coupled first order differential equations in the variable t . Similarly, (17) gives rise to $M \times N \times P$ coupled first order differential equations in the variable u . Each of (16),(17) then has $M \times N \times P$ linearly independent solutions.

Consider the function

$$\mathbf{g}(t, u, J, K, L) = \mathbf{C}(J, K, L) t^{-h_1-h_2+h_3-J-K+L} u^{-h_2-h_3+h_1+J-K-L} \times (u + t)^{-h_1-h_3+h_2-J+K-L} \quad (18)$$

On expanding $\mathbf{C}(J, K, L)$ in J, K, L , one can see that \mathbf{C} has $M \times N \times P$ components. By direct substitution, \mathbf{g} satisfies each of (16) and (17). Since \mathbf{g} has $M \times N \times P$ free components, one can conclude that it is the most general expression for the solution of (16) and (17). Note that there are no conditions on \mathbf{C} , nor are there any conditions on M, N, P .

These results appear to be in agreement with the literature, e.g. after restricting (18) to the rank two case, and the case of primaries not being pre-logarithmic, the results here match the results of [12].

4 Four point function

First consider the ‘usual’ case without Jordan blocks. One wishes to calculate

$$G^{(4)}(z_1, z_2, z_3, z_4) = \langle \phi_1(z_1) \phi_2(z_2) \phi_3(z_3) \phi_4(z_4) \rangle \quad (19)$$

where the ϕ_i are quasiprimary fields. The J_i are nilpotent, although they need not be nilpotent of the same degree, i.e. they satisfy $J_i^{r_i} = 0$, $J_i^{r_i-1} \neq 0$ where the r_i are need not be the same. In order to perform the calculation, consider the change of co-ordinates

$$u = (z_1 - z_2), \quad v = (z_2 - z_3), \quad x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}, \quad t = (z_1 + z_4). \quad (20)$$

The Ward identity for L_{-1} then reads

$$\sum_{i=1}^4 \frac{\partial}{\partial z_i} G^{(4)} = 2 \frac{\partial}{\partial t} G^{(4)} = 0 \quad (21)$$

and hence $G^{(4)} = G^{(4)}(u, v, x)$. Defining $H = \frac{1}{3} \sum_{i=1}^4 h_i$, the Ward identity for L_0 reads, after using (21),

$$\left(\sum_{i=1}^4 h_i + z_i \frac{\partial}{\partial z_i} \right) G^{(4)} = \left(3H + u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} \right) G^{(4)} = 0. \quad (22)$$

The L_1 Ward identity, after use of (21) and (22) reads

$$\begin{aligned} \sum_{i=1}^4 \left(2h_i z_i + z_i^2 \frac{\partial}{\partial z_i} \right) G^{(4)} = & \left[(h_1 + h_2 + h_3 - h_4) \left(\frac{uv}{u - x(u+v)} - v \right) + \right. \\ & (h_1 - h_2 - h_3 - h_4)u + (h_1 + h_2 - h_3 - h_4)v + \\ & \left. \left(\frac{uv}{u - x(u+v)} - v \right) \left(u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} \right) + uv \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v} \right) \right] G^{(4)} = 0. \end{aligned} \quad (23)$$

Defining

$$\begin{aligned} F(x, u, v) = G^{(4)} u^{h_1+h_2-H} v^{h_2+h_3-H} (u+v)^{h_1+h_3-H} & \left(\frac{uv}{u - x(u+v)} - v \right)^{h_3+h_4-H} \times \\ & \left(\frac{uv}{u - x(u+v)} \right)^{h_2+h_4-H} \left(\frac{uv}{u - x(u+v)} + u \right)^{h_1+h_4-H} \end{aligned} \quad (24)$$

one finds that (22) reduces to

$$\frac{G^{(4)}}{F} \left(u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} \right) F = 0 \quad (25)$$

and, after use of (25), that (23) reduces to

$$\frac{G^{(4)}}{F} uv \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v} \right) F = 0. \quad (26)$$

Thus, when the points z_i are not coincident, one has

$$\frac{\partial}{\partial u} F = 0 = \frac{\partial}{\partial v} F \quad (27)$$

and hence $F = F(x)$. After reorganizing the factors, the general four point function is then given by (24). In the logarithmic case, one wishes to consider the four point function

$$\mathbf{G}^{(4)}(z_1, z_2, z_3, z_4, J_1, J_2, J_3, J_4) = \langle \phi_1(z_1, J_1) \phi_2(z_2, J_2) \phi_3(z_3, J_3) \phi_4(z_4, J_4) \rangle \quad (28)$$

where the ϕ_i are logarithmic quasiprimary. Similar to the usual case, $\frac{\partial}{\partial t}\mathbf{G}^{(4)} = 0$. One must now define $H = \frac{1}{3}\sum_{i=1}^4 h_i + J_i$, define

$$\begin{aligned} \mathbf{F}(x, u, v, J_i) = & \mathbf{G}^{(4)} u^{h_1+h_2+J_1+J_2-H} v^{h_2+h_3+J_2+J_3-H} (u+v)^{h_1+h_3+J_1+J_3-H} \times \\ & \left(\frac{uv}{u-x(u+v)} - v \right)^{h_3+h_4+J_3+J_4-H} \left(\frac{uv}{u-x(u+v)} \right)^{h_2+h_4+J_2+J_4-H} \times \\ & \left(\frac{uv}{u-x(u+v)} + u \right)^{h_1+h_4+J_1+J_4-H} \end{aligned} \quad (29)$$

and use the Ward identities

$$\begin{aligned} & \left(\sum_{i=1}^4 h_i + J_i + z_i \frac{\partial}{\partial z_i} \right) \mathbf{G}^{(4)} = 0 \\ & \sum_{i=1}^4 \left(2(h_i + J_i) z_i + z_i^2 \frac{\partial}{\partial z_i} \right) \mathbf{G}^{(4)} = 0 \end{aligned} \quad (30)$$

in a similar manner to the usual case to deduce that $\mathbf{F} = \mathbf{F}(x, J_i)$. Thus one finds that

$$\begin{aligned} \mathbf{G}^{(4)}(u, v, x, J_i) = & \mathbf{F}(x, J_i) u^{-h_1-h_2-J_1-J_2+H} v^{-h_2-h_3-J_2-J_3+H} \times \\ & (u+v)^{-h_1-h_3-J_1-J_3+H} \left(\frac{uv}{u-x(u+v)} - v \right)^{-h_3-h_4-J_3-J_4+H} \times \\ & \left(\frac{uv}{u-x(u+v)} \right)^{-h_2-h_4-J_2-J_4+H} \left(\frac{uv}{u-x(u+v)} + u \right)^{-h_1-h_4-J_1-J_4+H} \end{aligned} \quad (31)$$

which reads in the original co-ordinates (where x is the cross-ratio)

$$\mathbf{G}^{(4)}(z_i, J_i) = \mathbf{F}(x, J_i) \prod_{i < k} (z_i - z_k)^{-h_i-h_k-J_i-J_k+H} \quad (32)$$

is the most general logarithmic four point function permitted by Möbius symmetry. Note that since the Jordan blocks satisfy $J_i^{r_i} = 0$, $J_i^{r_i-1} \neq 0$, then \mathbf{F} represents $r_1 r_2 r_3 r_4$ functions of cross-ratios x . On expanding into components, $\mathbf{G}^{(4)}$ will contain logarithms that mix the components of \mathbf{F} amongst the various individual four point functions.

It is instructive to compare this result to examples in the literature[2], where the actual primary fields in a Jordan block are not pre-logarithmic. \mathbf{F} can be expanded as $\mathbf{F}(x) = F_0(x) + \sum_{i=1}^4 J_i F_i(x) \dots$. Taking $F_0 = 0$ and $F_1 = F_2$, one finds that

$$G_{12} = \frac{1}{3} \left(\prod_{i < j} z_{ij}^{\mu_{ij}} \right) \left(3F_{12} + F_1(-2l_{12} + l_{13} + l_{14} - 2l_{23} - 2l_{24} + l_{34}) \right)$$

$$\begin{aligned}
& +F_2(-2l_{12} - 2l_{13} - 2l_{14} + l_{23} + l_{24} + l_{34})) \\
& = \frac{1}{3} \left(\prod_{i < j} z_{ij}^{\mu_{ij}} \right) \left(3F_{12} + F_1 \left(-6l_{12} + \log(x) + \log \left(\frac{x}{1-x} \right) \right) \right) \quad (33)
\end{aligned}$$

where $z_{ij} = z_i - z_j$, $\mu_{ij} = -h_i - h_j + \frac{1}{3} \sum_{k=1}^4 h_k$ and $l_{ij} = \log(z_i - z_j)$. Thus, one finds that logarithms of the cross ratio can appear.

5 $c = 0$ disordered systems

One starts with a Virasoro OPE, with vanishing central charge

$$T(z)T(w) = \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \dots \quad (34)$$

So as not to let $L_{-2}|0\rangle = |T\rangle$ decouple, leaving a trivial theory, one can try to realize the theory with a logarithmic partner field

$$T(z)t(w) = \frac{b}{(z-w)^4} + \frac{2t(w) + T(w)}{(z-w)^2} + \frac{\partial t(w)}{(z-w)} + \dots \quad (35)$$

where b is some undefined constant. Note, that since $\langle T(z)t(w) \rangle \neq 0$ is required, this implies that $\langle 0|0 \rangle \neq 0$. From comparing with the two point function (10) of a logarithmic quasiprimary field, this implies that the identity operator cannot be a part of a Jordan block. Hence, there are three fundamental fields in the theory; the identity 1, the Virasoro generator T , and the Virasoro generator's logarithmic partner t . For this system (34), (35) and the field content of $\{1, T, t\}$ will be the only facts assumed. One can then try and construct an operator algebra, and ask if that algebra is consistent. One can immediately read off the two point functions

$$\langle T(z)t(w) \rangle = \frac{b}{(z-w)^4}, \quad \langle T(z)T(w) \rangle = 0 \quad (36)$$

and notice that $|T\rangle$ has a non-trivial inner product with $|t\rangle$, and hence cannot decouple. One can then use global conformal symmetry transformations on $\langle t(z)t(w) \rangle$, which can be obtained from (35), to deduce

$$\langle t(z)t(w) \rangle = \frac{e - 2b \log(z-w)}{(z-w)^4} \quad (37)$$

where e is an arbitrary constant. Since the generic form of the two and three point functions are known from global conformal symmetry, one might ask about the operator content of the theory, in a similar manner that one does for ordinary CFT. Requiring the fields to form a closed, associative, commutative operator algebra usually imposes constraints. From (37), it can be seen that what are normally structure constants will now become functions. A similar statement holds for the three point functions. One can denote the structure functions, C , as

$$\phi_i(x)\phi_j(y) = \frac{C_{ij}{}^k(x, y)\phi_k(y)}{(x - y)^{h_i+h_j-h_k}} + \dots \quad (38)$$

where, as usual, \dots represent terms with poles in $(x - y)$ of order less than $h_i + h_j - h_k$. Similarly, the ‘coefficient’ in front of a three point function can be denoted $C_{ijk}(x, y, z)$. By Taylor expanding (38), one has

$$\phi_i(x)\phi_j(y) = \frac{C_{ij}{}^k(x, y)\phi_k(x)}{(x - y)^{h_i+h_j-h_k}} + \dots \quad (39)$$

Thus, demanding commutativity of the operator algebra, requires

$$C_{ij}{}^k(x, y) = C_{ji}{}^k(y, x). \quad (40)$$

Already, from (37), this can be seen to be too strong a constraint to impose on all of the structure functions. However, some of the structure functions do exhibit commutativity, in particular those that are constant. Labelling the fields $\{1, T, t\}$ as $\{\phi_1, \phi_2, \phi_3\}$, one can see from the OPEs (34), (35), that

$$C_{22}{}^1 = 0, \quad C_{22}{}^2 = 2, \quad C_{22}{}^3 = 0, \quad C_{23}{}^1 = b, \quad C_{23}{}^2 = 1, \quad C_{23}{}^3 = 2, \quad C_{1j}{}^k = \delta_j^k \quad (41)$$

where the $C_{ij}{}^k$ are symmetric in i, j . From the two point functions (37), one has

$$C_{33}{}^1(x, y) = e - 2b \log(x - y) \quad (42)$$

which represents a structure function not obeying (40). Indeed, the product $t(z)t(w)$ is the only offending product against commutativity.

Now, using (34) and (35), one can see that $T(w) + Jt(w) =: T(J, w)$ is a quasiprimary logarithmic field, and hence its three point function is given by (18). From (35) and using

the two point functions, one can deduce that

$$\begin{aligned} \lim_{|z-w| \rightarrow 0} \langle T(z)t(w)t(u) \rangle &= \lim_{|z-w| \rightarrow 0} \frac{C_{233}(z, w, u)}{(z-w)^2(w-u)^2(z-u)^2} \\ &= \lim_{|z-w| \rightarrow 0} \frac{b + 2e - 4b \log(w-u)}{(z-w)^2(w-u)^4} + O((z-w)^{-1}). \end{aligned} \quad (43)$$

One can do the same with the $\langle T(z)T(w)T(u) \rangle$ and $\langle T(z)T(w)t(u) \rangle$ correlators. Now, comparing with (18), it can be seen that the $C_{233}(z, w, u)$, $C_{223}(z, w, u)$ and $C_{222}(z, w, u)$ found are in fact the most general, even away from $|z-w| \rightarrow 0$. Also, since T commutes with t , this result also yields C_{323} and C_{332} . Similarly, $C_{223} = 2b = C_{232} = C_{322}$, $C_{222} = 0$. Using these numbers, and the general form of the 3 point function (18), one can deduce that

$$\langle t(x)t(y)t(z) \rangle = \frac{C_{333}(x, y, z)}{(x-y)^2(y-z)^2(x-z)^2} \quad (44)$$

where

$$\begin{aligned} C_{333}(x, y, z) &= d - (b + 2e) \left(\log(x-z) + \log(x-y) + \log(y-z) \right) \\ &\quad - 2b \left(\log^2(x-z) + \log^2(x-y) + \log^2(y-z) - 2 \log(x-y) \log(x-z) \right. \\ &\quad \left. - 2 \log(x-y) \log(y-z) - 2 \log(y-z) \log(x-z) \right). \end{aligned} \quad (45)$$

Now,

$$t(x)t(y) = \frac{C_{33}^1(x, y)1}{(x-y)^4} + \frac{C_{33}^2(x, y)T(y)}{(x-y)^2} + \frac{C_{33}^3(x, y)t(y)}{(x-y)^2} + \dots \quad (46)$$

where \dots represents terms with at most a simple pole in $x-y$. Thus

$$\langle t(x)t(y)T(z) \rangle = \frac{C_{33}^3(x, y)\langle t(y)T(z) \rangle}{(x-y)^2} + \dots \quad (47)$$

Considering the limit $|x-y| \rightarrow 0$, and using $b \neq 0$, one can deduce that

$$C_{33}^3(x, y) = 1 + 2\frac{e}{b} - 4 \log(x-y). \quad (48)$$

Similarly,

$$\langle t(x)t(y)t(z) \rangle = \frac{C_{33}^2(x, y)\langle T(y)t(z) \rangle}{(x-y)^2} + \frac{C_{33}^3(x, y)\langle t(y)t(z) \rangle}{(x-y)^2} + \dots \quad (49)$$

and the limit $|x - y| \rightarrow 0$ yields

$$C_{33}^2(x, y) = \frac{1}{b} \left(d - e \left(1 + \frac{2e}{b} \right) - (b - 2e) \log(x - y) - 2b \log^2(x - y) \right) \quad (50)$$

and hence all of the structure constants are obtained.

Given the field content of the theory, the most general singular terms that can appear in the $t(x)t(y)$ OPE are given by

$$\begin{aligned} t(x)t(y) = & \frac{C_{33}^1(x, y)1}{(x - y)^4} + \frac{C_{33}^2(x, y)T(y)}{(x - y)^2} + \frac{C_{33}^3(x, y)t(y)}{(x - y)^2} + \\ & \frac{A(x, y)\partial t(y)}{(x - y)} + \frac{B(x, y)\partial T(y)}{(x - y)} + \dots \end{aligned} \quad (51)$$

One can use conformal ‘invariance’ of the theory to obtain A and B , i.e. note

$$[L_1, t(x)t(y)]|0\rangle = [L_1, t(x)]t(y)|0\rangle + t(x)[L_1, t(y)]|0\rangle \quad (52)$$

and one can take the OPE and act with L_1 , or act with L_1 then take the OPE. Comparing the T and t coefficients from these two calculations yields

$$A = \frac{1}{2} \left(1 + \frac{2e}{b} - 4 \log(x - y) \right) = \frac{1}{2} C_{33}^3 \quad (53)$$

$$B = \frac{1}{2b} \left(d - e \left(1 + \frac{2e}{b} \right) - (b - 2e) \log(x - y) - 2b \log^2(x - y) \right) = \frac{1}{2} C_{33}^2 \quad (54)$$

and thus all singular components of the OPE are known.

These calculations give the operator algebra found in [11], where the algebra was found by different methods. In particular, since in this note only (34) and (35) were needed to obtain the operator algebra, the above calculations answer an important question - given a $c = 0$ system with non-degenerate vacuum, where the Virasoro generator T has a logarithmic partner, does one always arrive at the same operator algebra? Up to parameters b, d (e can be removed by redefinition of t - see next section), the answer is yes. Note that $b \neq 0$ has been used, which is necessary for T not to decouple, as was the original motivation. If one wishes to consider $b = 0$, and T not decoupling, then from looking at the two point function (8), larger rank Jordan blocks will be needed.

6 Associativity

There are a number of free parameters, namely b, e, d . For $b \neq 0$, t can be redefined by $t \mapsto t - \frac{e}{2b}T$ which leaves (35) unchanged. However, it does set $e = 0$ in (37). Whilst

strictly, one needs to look at the four-point function to understand the constraints arising from associativity, by looking at an analogous algebraic structure, one can formally solve for the constraints. Consider an associative algebra spanned by a finite number of ‘fields’ $\{A_I\}$, over the polynomial ring $\mathbb{C}[x]$. Multiplication is given by

$$A_I A_J = \sum_P C_{IJ}^P(x) A_P. \quad (55)$$

This mimics the operator algebra structure, with the logarithms given by x . Associativity of this algebra imposes

$$\sum_P C_{IJ}^P(x) C_{PK}^L(x) = \sum_P C_{IP}^L(x) C_{JK}^P(x) \quad (56)$$

on the structure constants. One can now try to impose this structure on the algebra at hand

$$\sum_p C_{ij}^p(x) C_{pk}^l(x) = \sum_p C_{ip}^l(x) C_{jk}^p(x). \quad (57)$$

Requiring associativity, and setting $(i, j, k, l) = (2, 2, 3, 2)$ in (57) and using only (41) which were assumed at the beginning of the calculation, yields $b + 2 = 0$. One can check that for $b + 2 = 0$, the identity holds for all (i, j, k, l) .

One can ask if this result can be reproduced using the four point function. Since T has its only non-zero two point function when the other field in the correlator is t , $(i, j, k, l) = (2, 2, 3, 2)$ is equivalent to studying the four point function $\langle T(x)T(y)t(w)t(z) \rangle$. Now, $T(x)$ has a mode expansion $T(x) = \sum_{m \in \mathbb{Z}} L_m z^{-m-2}$. Given this mode expansion, (34) and (35) then yield

$$[L_m, T(z)] = 2(m+1)z^m T(z) + z^{m+1} \partial T(z) \quad (58)$$

$$[L_m, t(z)] = \frac{b}{6} m(m^2 - 1) z^{m-2} + (m+1) z^m (2t(z) + T(z)) + z^{m+1} \partial t(z). \quad (59)$$

Using these commutation relations and

$$\langle 0 | L_m = 0 \text{ for } m \leq 1, \quad L_n | 0 \rangle = 0 \text{ for } n \geq -1, \quad (60)$$

one can calculate $\langle T(x)T(y)t(w)t(z) \rangle$ for $|x| > |y|$ in terms of two and three point functions, and analytically continue in $(x - y)$ to obtain the full four point function.

This yields

$$\begin{aligned}
\langle T(x)T(y)t(w)t(z) \rangle &= \frac{2\langle T(y)t(w)t(z) \rangle}{(x-y)^2} + \frac{\langle \partial T(y)t(w)t(z) \rangle}{(x-y)} + \frac{b\langle T(y)t(z) \rangle}{(x-w)^4} + \\
&\quad \frac{2\langle T(y)t(w)t(z) \rangle}{(x-w)^2} + \frac{\langle T(y)T(w)t(z) \rangle}{(x-w)^2} + \frac{\langle T(y)\partial t(w)t(z) \rangle}{(x-w)} + \\
&\quad \frac{b\langle T(y)t(w) \rangle}{(x-z)^4} + \frac{2\langle T(y)t(w)t(z) \rangle}{(x-z)^2} + \frac{\langle T(y)t(w)T(z) \rangle}{(x-z)^2} + \\
&\quad \frac{\langle T(y)t(w)\partial t(z) \rangle}{(x-z)}. \tag{61}
\end{aligned}$$

The identity in question comes down to taking the OPE of $T(x)$ with $T(y)$ and then evaluating the four point function, and comparing this to taking the $T(y)t(w)$ OPE and evaluating the four point function. To this end, consider

$$\lim_{|x-y| \rightarrow 0} \langle T(x)T(y)t(w)t(z) \rangle = \lim_{|x-y| \rightarrow 0} \left\langle \left(\frac{2T(y)}{(x-y)^2} + \frac{\partial T(y)}{(x-y)} \right) t(w)t(z) \right\rangle + O((x-y)^0) \tag{62}$$

which, in the limit $|x-y| \rightarrow 0$, agrees with (61). Similarly, consider

$$\begin{aligned}
\lim_{|y-w| \rightarrow 0} \langle T(x)T(y)t(w)t(z) \rangle &= \lim_{|y-w| \rightarrow 0} \langle T(x) \left(\frac{b}{(y-w)^4} + \frac{2t(w) + T(w)}{(y-w)^2} + \frac{\partial t(w)}{(y-w)} \right) t(z) \rangle \\
&\quad + O((y-w)^0). \tag{63}
\end{aligned}$$

This does not obviously agree with (61). However, taking $|y-w| = |x-z| = \epsilon$ and using the expressions for the three point functions (or, in the limit using, the $T(x)t(z)$ OPE and the two point functions), one can show that as $\epsilon \rightarrow 0$, both (63) and (61) yield

$$\lim_{\epsilon \rightarrow 0} \frac{b^2}{(y-w)^4(x-z)^4} + \frac{2(b+2e-4b \log(w-z)) + 2b}{(y-w)^2(x-z)^2(x-w)^2(w-z)^2} + O(\epsilon^{-3}) \tag{64}$$

and hence, in the limit, both functions agree. In particular, there is no restriction on b . Hence, the notion of $b+2=0$ is really an illusion from performing too naïve a calculation, and missing out the conformal blocks in (57). However, since naïvely the only thing stopping associativity seemed to be $b+2 \neq 0$, one might suspect that the algebra is in fact associative for arbitrary b . Of course, to check this properly would require checking all of the four point functions, but since $t(z)$ does not appear to have a mode expansion, it is not obvious to the author how to compute $\langle t(x)t(y)t(w)t(z) \rangle$. However, using the mode expansion of T and the three point functions, all the other four point functions can be calculated, and can be used to check associativity. Unfortunately, as can be seen from the general form of the four point function obtained from only Möbius

symmetry (32), the calculable four point functions are not quite enough to obtain the $\langle tttt \rangle$ correlation function - there is still one arbitrary function of the cross-ratio that needs to be found. As such, a full calculation to check the associativity of the operator algebra is still an open problem. Nonetheless, one can check the calculable four point functions to see if any yield non-associativity (as done in the appendix), and they do not. In particular, they do not give any constraints on b .

Thus, assuming just (34), (35), one can deduce a general operator algebra, which one might expect to be associative, which has two free components, namely b, d .

7 Conclusions

Purely from the Ward identities for Möbius symmetry, the general three and four point functions were obtained. Whilst the author has not done it, one should be able to find the higher N point functions by a similar calculation, with the change of co-ordinates involving more cross-ratios. If the primary fields in the Jordan blocks of the logarithmic primaries are not pre-logarithmic, then further constraints appear on the three and four point functions, which amounts to setting some constants to zero in the three point case, and some functions of cross-ratios to zero in the four point case.

Taking the example of $c = 0$ systems with non-degenerate vacua, the three point function proved to be extremely useful in finding the operator algebra, and the four point function was useful in the analysis of associativity.

In the analysis of $c = 0$ systems, it was found that just assuming that T had a logarithmic partner and that T did not decouple, one could deduce that the identity was not a member of a Jordan block, and deduce the full operator algebra, which was parameterized by two constants, b and d . This result matches that of [11], although the derivation here is different, and possibly more general. On a formal level, the associativity conditions were checked. On a more precise level, almost all of the associativity conditions were checked. Since the author was unable to obtain the $\langle tttt \rangle$ correlator, it still remains an open question as to whether or not this four point function yields any conditions on associativity. However, all the other four point functions could be found, and were tested to see if they gave signs of non-associativity. They did not. These findings suggest that $c = 0$ systems with non-degenerate vacua may well give consistent field theories, although the final steps of the argument remain unfinished.

The operator algebra obtained differed quite significantly to those in normal CFT, in that due to logarithms in the OPE, it is not obvious how to relate $t(z)t(w)$ to $t(w)t(z)$. One resolution might be to define $\bar{\partial}t(z, \bar{z})$ as an antiholomorphic weight 1 primary field,

similar to a free boson, as touched on in [10]. The logarithms would then appear as $\log |z - w|$ rather than $\log(z - w)$, and hence $t(z)t(w)$ might be symmetric. The logical end to this input, and the resultant operator algebra, is not something the author has done.

The author hopes that this note has given a good illustration of how BPZ [1] machinery can be generalized to the logarithmic scenario.

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Note added

During the writing of this manuscript, [14] was released, where, in the case of rank two Jordan blocks and without assuming anything about the operator algebra, the three point function was obtained. After restricting the three point function (18) to the rank two case, (18) then agrees with the three point function found in [14].

A $c=0$ four point functions and associativity

Since if there is more than one identity operator in correlator, the calculation will clearly give an answer of associativity, only the case of ≤ 1 operator in the correlator being the identity will be considered. Since $C_{22}^3 = C_{22}^1 = 0$, and the correlators $\langle TT \rangle$, $\langle TTT \rangle$ and $\langle TTTT \rangle$ are zero, the correlators $\langle TTT1 \rangle$ and $\langle TTTT \rangle$ will yield associativity. Checking that the $\langle ttt1 \rangle$, $\langle Ttt1 \rangle$ and $\langle TTt1 \rangle$ correlators yield associativity are not difficult or long calculations, and indeed they do.

Using (58), (59), (60), as before, one can compute $\langle T(x)T(y)T(z)t(w) \rangle$ to give (after noting the $\langle TT \rangle$ and $\langle TTT \rangle$ correlators are zero)

$$\begin{aligned} \langle T(x)T(y)T(z)t(w) \rangle = & \frac{2\langle T(y)T(z)t(w) \rangle}{(x-y)^2} + \frac{\langle \partial T(y)T(z)t(w) \rangle}{(x-y)} + \frac{2\langle T(y)T(z)t(w) \rangle}{(x-z)^2} + \\ & \frac{\langle T(y)\partial T(z)t(w) \rangle}{(x-z)} + \frac{2\langle T(y)T(z)t(w) \rangle}{(x-w)^2} + \frac{\langle T(y)T(z)\partial t(w) \rangle}{(x-w)} \end{aligned} \quad (65)$$

which clearly agrees with the $|x - y| \rightarrow 0$ limit and the $T(x)T(y)$ OPE

$$\lim_{|x-y| \rightarrow 0} \langle T(x)T(y)T(z)t(w) \rangle = \lim_{|x-y| \rightarrow 0} \left\langle \left(\frac{2T(y)}{(x-y)^2} + \frac{\partial T(y)}{(x-y)} \right) T(w)t(z) \right\rangle + O((x-y)^0). \quad (66)$$

Considering $|y - z| = |x - w| = \epsilon$ and $\epsilon \rightarrow 0$, both (65) and using the $T(y)T(z)$ OPE with three point functions, i.e.,

$$\lim_{|y-z| \rightarrow 0} \langle T(x)T(y)T(z)t(w) \rangle = \lim_{|y-z| \rightarrow 0} \langle T(x) \left(\frac{2T(z)}{(y-z)^2} + \frac{\partial T(z)}{(y-z)} \right) t(w) \rangle + O((y-z)^0) \quad (67)$$

yield

$$\lim_{\epsilon \rightarrow 0} \langle T(x)T(y)T(z)t(w) \rangle = \lim_{\epsilon \rightarrow 0} \frac{4b}{(x-w)^2(y-z)^2(z-w)^2(x-z)^2} + O(\epsilon^{-3}) \quad (68)$$

and hence agree in this limit.

Using the same techniques, the $\langle Tttt \rangle$ correlator is given by

$$\begin{aligned} \langle T(x)t(y)t(w)t(z) \rangle &= \frac{b\langle t(w)t(z) \rangle}{(x-y)^4} + \frac{2\langle t(y)t(w)t(z) \rangle + \langle T(y)t(w)t(z) \rangle}{(x-y)^2} + \frac{\langle \partial t(y)t(w)t(z) \rangle}{(x-y)} + \\ &\frac{b\langle t(y)t(z) \rangle}{(x-w)^4} + \frac{2\langle t(y)t(w)t(z) \rangle + \langle t(y)T(w)t(z) \rangle}{(x-w)^2} + \frac{\langle t(y)\partial t(w)t(z) \rangle}{(x-w)} + \\ &\frac{b\langle t(y)t(w) \rangle}{(x-z)^4} + \frac{2\langle t(y)t(w)t(z) \rangle + \langle t(y)t(w)T(z) \rangle}{(x-z)^2} + \frac{\langle t(y)t(w)\partial t(z) \rangle}{(x-z)} \end{aligned} \quad (69)$$

which in the limit $|x - y| \rightarrow 0$ agrees with the OPE

$$\lim_{|x-y| \rightarrow 0} \langle T(x)t(y)t(w)t(z) \rangle = \lim_{|x-y| \rightarrow 0} \left\langle \left(\frac{b}{(x-y)^4} + \frac{2t(y) + T(y)}{(x-y)^2} + \frac{\partial t(y)}{(x-y)} \right) t(w)t(z) \right\rangle + O((x-y)^0). \quad (70)$$

To analyze the other OPE, one must once again set $|y - w| = |x - z| = \epsilon$ and take the limit and OPE. In this case, once again the four point function and OPE calculation agree, with leading order behaviour

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{b(e - 2 \log(y - w))}{(y - w)^4(x - z)^4} &+ \frac{b + 2e + 2d - (4b + 8e) \log(y - x) - (4e + 6b) \log(y - w)}{(y - w)^2(x - z)^2(x - y)^4} \\ &+ \frac{16b \log(y - x) \log(y - w) - 4b \log^2(y - w)}{(y - w)^2(x - z)^2(x - y)^4} + o(\epsilon^{-3-\frac{1}{2}}) \end{aligned} \quad (71)$$

where the $\epsilon^{-\frac{1}{2}}$ is to suppress the logs in the limit.

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